# Applications of the Mittag-Leffler function

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## 1. INTRODUCTION AND DEFINITIONS

Let $\mathbf{E}_\alpha$ be the function defined by

$$\mathbf{E}_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)},\ z \in \mathbb{C}, \alpha \in \mathbb{C}, \text{ with } \operatorname{Re}\alpha > 0$$

that was introduced by Mittag-Leffler [14] and commonly known as the Mittag-Leffler function. Wiman [25] defined a more general function $\mathbf{E}_{\alpha,\beta}$ generalizing the $\mathbf{E}_\alpha$ Mittag-Leffler function, that is

$$\mathbf{E}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)},\ z \in \mathbb{C}, \alpha, \beta \in \mathbb{C}, \text{ with } \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0$$

When $\beta = 1$, it is abbreviated as $\mathbf{E}_\alpha(z) = \mathbf{E}_{\alpha,1}(z)$. Observe that the function $\mathbf{E}_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$\mathbf{E}_{1,1}(z) = e^z,\ \mathbf{E}_{1,2}(z) = \frac{e^z - 1}{z},\ \mathbf{E}_{2,1}(z^2) = \cosh z$$
$$\mathbf{E}_{2,1}(-z^2) = \cos z,\ \mathbf{E}_{2,2}(z^2) = \frac{\sinh z}{z},\ \mathbf{E}_{2,2}(-z^2) = \frac{\sin z}{z},$$
$$\mathbf{E}_4(z) = \frac{1}{2}\left(\cos z^{1/4} + \cosh z^{1/4}\right),\ \mathbf{E}_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right].$$

We recall the error function erf given by [1, p. 297]

$$\operatorname{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(2n+1)}z^{2n+1}$$

the complement of the error function erfc defined by

$$\operatorname{erfc}(z) := 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(2n+1)}z^{2n+1},$$

and the normalized form of the error function erf denoted by Erf (normalized with the condition $\operatorname{Erf}'(0) = 1$) is given by

$$\operatorname{Erf}(z) := \frac{\sqrt{\pi z}}{2}\operatorname{erf}(\sqrt{z}) = z + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!\,(2n-1)}z^n$$

It is of interest to note that by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2}\mathrm{erfc}(-z),$$

that is

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2}\left(1 + \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} z^{2n+1}\right).$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found for example in [2,3,8,9,11,12]. We note that the above generalized (Mittag-Leffler) function $\mathbf{E}_{\alpha,\beta}$ does not belongs to the family $\mathcal{A}$, where $\mathcal{A}$ represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathbb{D}, (1.1)$$

Prabhakar [26, 27] has generalized the Mittag – Leffler function as follows

$$E_{\alpha,\beta}^{\theta}(z) := \sum_{n=0}^{\infty} \frac{(\theta)_n}{\Gamma(\alpha n + \beta)}.\ \frac{z^n}{n!},\quad z, \beta, \theta\ \in\ \mathbb{C}\,;\ \mathrm{Re}\,\alpha > 0.$$

Note that $(\theta)_v$ denotes the familiar Pochhammer symbol defined as

$$(\theta)_v := \frac{\Gamma(\theta + v)}{\Gamma(\theta)} = \begin{cases} 1, if\ v = 0, & \theta \in\ \mathbb{C}\backslash\{0\} \\ \theta(\theta+1)\ldots(\theta+n-1), if\ v = n\ \in N, \theta \in\ \mathbb{C} \end{cases}$$

$$(1)_n = n!\ n\ \in N_0, N_0 = N\ \cup \{0\}, \qquad N = \{1,2,3,\ldots\}$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{D}$. Thus, it is expected to define the following normalization of Mittag-Leffler functions as below, due to Bansal and Prajapat [3]:

$$E_{\alpha,\beta}^{\theta}(z) := z\Gamma(\beta)\mathbf{E}_{\alpha,\beta}^{\theta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)\,(\theta)_n}{\mathrm{n!}\,\Gamma(\alpha(n-1)+\beta)} z^n (1.2)$$

that holds for the parameters $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0, \mathrm{Re}\,\beta > 0$ and $z \in \mathbb{C}$. In this paper we shall confine our attention to the case of real-valued parameters $\alpha$ and $\beta$, and we will consider that $z \in \mathbb{D}$.

For functions $f \in \mathcal{A}$ be given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$, we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in \mathbb{D}.$$

The two well-known subclasses of $\mathcal{S}$ are namely the class of starlike and convex functions (for details see Robertson [20]). Thus, a function $f \in \mathcal{A}$ given by (1.1) is said to be starlike of order $\gamma, 0 \leq \gamma < 1$, if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \ z \in \mathbb{D},$$

and this function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike domain with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order $\gamma, 0 \leq \gamma < 1$, if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \ z \in \mathbb{D}$$

and this class is denoted by $\mathcal{K}(\gamma)$. Further, $\mathcal{K} := \mathcal{K}(0)$ represents the well-known standard class of convex functions. By Alexander's duality relation (see [6]), it is a known fact that

$$f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

A function $f \in \mathcal{A}$ is said to be spiral-like if

$$\operatorname{Re}\left(e^{-i\xi} \frac{zf'(z)}{f(z)}\right) > 0, \ z \in \mathbb{D},$$

for some $\xi \in \mathbb{C}$ with $|\xi| < \frac{\pi}{2}$, and the class of spiral-like functions was introduced in [23]. Also, the function $f$ is said to be convex spiral-like if $zf'(z)$ is spiral-like. Due to Murugusundramoorthy [15,16], we consider the following subclasses of spiral-like functions as below.

Definition 1.1. For $0 \leq \rho < 1, 0 \leq \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{S}(\xi, \gamma, \rho)$ by

$$\mathcal{S}(\xi, \gamma, \rho) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\xi} \frac{zf'(z)}{(1-\rho)f(z) + \rho zf'(z)}\right) > \gamma \cos \xi, z \in \mathbb{D} \right\}$$

By virtue of Alexander's relation (see [6]) we define the following subclass $\mathcal{K}(\xi, \gamma, \rho)$.

Definition 1.2. For $0 \leq \rho < 1, 0 \leq \gamma < 1$ and $|\xi| < \frac{\pi}{2}$, let define the class $\mathcal{K}(\xi, \gamma, \rho)$ by

$$\mathcal{K}(\xi, \gamma, \rho) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\xi} \frac{zf''(z) + f'(z)}{f'(z) + \rho zf''(z)}\right) > \gamma \cos \xi, z \in \mathbb{D} \right\}.$$

By specializing the parameter $\rho = 0$ in the above two definitions we obtain the subclasses $\mathcal{S}(\xi, \gamma) := \mathcal{S}(\xi, \gamma, 0)$ and $\mathcal{K}(\xi, \gamma) := \mathcal{K}(\xi, \gamma, 0)$, respectively.

Now we state a sufficient conditions for the function $f$ to be in the above classes.

Lemma 1.1([15,16]). A function $f$ given by (1.1) is a member of $\mathcal{S}(\xi,\gamma,\rho)$ if

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]|a_n|\leq 1-\gamma$$

where $|\xi|<\frac{\pi}{2}, 0\leq\rho<1, 0\leq\gamma<1$

Since $f\in\mathcal{K}(\xi,\gamma,\rho)$ if and only if $zf'(z)\in\mathcal{S}(\xi,\gamma,\rho)$, and from Lemma 1.1 we get the next result.

Lemma 1.2. A function $f$ given by (1.1) is a member of $\mathcal{K}(\xi,\gamma,\rho)$ if

$$\sum_{n=2}^{\infty}n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]|a_n|\leq 1-\gamma,$$

where $|\xi|<\frac{\pi}{2}, 0\leq\rho<1, 0\leq\gamma<1$.

The next class $\mathcal{R}^{\tau}(\vartheta,\delta)$ was introduced earlier by Swaminathan [24], and for special cases see the references cited there in.

Definition 1.3. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(\vartheta,\delta)$, where $\tau\in\mathbb{C}\setminus\{0\}$, $0<\vartheta\leq 1$, and $\delta<1$, if it satisfies the inequality

$$\left|\frac{(1-\vartheta)\frac{f(z)}{z}+\vartheta f'(z)-1}{2\tau(1-\delta)+(1-\vartheta)\frac{f(z)}{z}+\vartheta f'(z)-1}\right|<1,\ z\in\mathbb{D}.$$

Lemma 1.3([24]). If $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$ is of the form (1.1), then

$$|a_n|\leq\frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)},\ n\in\mathbb{N}\setminus\{1\}(1.3)$$

The bounds given in (1.3) is sharp for

$$f(z)=\frac{1}{\vartheta z^{1-\frac{1}{\vartheta}}}\int_0^z t^{1-\frac{1}{\vartheta}}\left[1+\frac{2(1-\delta)\tau t^{n-1}}{1-2^{n-1}}\right]dt$$

Now we define the following new linear operator based on convolution (Hadamard) product. For real parameters $\alpha,\beta$, with $\alpha,\beta,\notin\{0,-1,-2,...\}$ and $E_{\alpha,\beta}$ be given by (1.2), we define the linear operator $\Lambda_{\beta}^{\alpha}:\mathcal{A}\to\mathcal{A}$ with the aid of the convolution product

$$\Lambda_{\beta}^{\alpha}f(z):=f(z)*E_{\alpha,\beta}^{\theta}(z)=z+\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}a_nz^n,\ z\in\mathbb{D}.$$

Stimulated by prior results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example [5,10,13,21,22, 24]) and by the

recent investigations related with distribution series (see for example $[4,7,17-19]$, we obtain sufficient condition for the function $E_{\alpha,\beta}^{\theta}$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions belonging in $\mathcal{R}^{\tau}(\vartheta,\delta)$ by using the convolution operator $\Lambda_{\beta}^{\alpha}$. Finally, we determined conditions for the integral operator $\Psi_{\beta}^{\alpha}(z)=\int_0^z \frac{E_{\alpha,\beta}^{\theta}(t)}{t}dt$ to belong to the above classes.

## 2. INCLUSION RESULTS

In order to prove our main results, unless otherwise stated throughout this paper, we will use the notation (1.2), therefore

$$E_{\alpha,\beta}^{\theta}(1)-1 = \sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}, \quad (2.1)$$

$$E^{\theta\prime}_{\alpha,\beta}(1)-1 = \sum_{n=2}^{\infty}\frac{n(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}, \quad (2.2)$$

$$E^{\theta\prime\prime}_{\alpha,\beta}(1) = \sum_{n=2}^{\infty}\frac{(\theta)_n n(n-1)\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}. \quad (2.3$$

Theorem 2.1. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]E_{\alpha,\beta}'(1)+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}^{\theta}(1)\le 2(1-\gamma), \quad (2.4)$$

then $E_{\alpha,\beta}^{\theta}\in\mathcal{S}(\xi,\gamma,\rho)$.

Proof. Since $E_{\alpha,\beta}$ is defined by (1.2), according to Lemma 1.1 it is sufficient to show that

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\le 1-\gamma \quad (2.5)$$

Since the left-hand side of the inequality (2.5) could be written as

$$\begin{aligned} Q_1(\xi,\gamma,\rho):= &\sum_{n=2}^{\infty}[(1-\rho)\sec\xi(n-1)+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)} \\ = &[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{n(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)} \\ &+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}, \end{aligned}$$

therefore, by using (2.1) and (2.2), we get

$$\begin{aligned} Q_1(\xi,\gamma,\rho) = \ & [(1-\rho)\sec\xi+\rho(1-\gamma)]\left[{E^{\theta}}'_{\alpha,\beta}(1)-1\right] \\ & +(1-\rho)(1-\gamma-\sec\xi)\left[E^{\theta}_{\alpha,\beta}(1)-1\right] \\ = \ & [(1-\rho)\sec\xi+\rho(1-\gamma)]{E^{\theta}}'_{\alpha,\beta}(1)+(1-\rho)(1-\gamma-\sec\xi)E^{\theta}_{\alpha,\beta}(1) \\ & -(1-\gamma) \end{aligned}$$

Thus, from the assumption (2.4) it follows that $Q_1(\xi,\gamma,\rho)\le 1-\gamma$, that is (2.5) holds, therefore $E^{\theta}_{\alpha,\beta}\in\mathcal{S}(\xi,\gamma,\rho)$.

Theorem 2.2. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]{E^{\theta}}''_{\alpha,\beta}(1)+(1-\gamma){E^{\theta}}'_{\alpha,\beta}(1)\le 2(1-\gamma), \quad (2.6)$$

then $E^{\theta}_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma,\rho)$.

Proof. Using the definition (1.2) of $E_{\alpha,\beta}$, in view of Lemma 1.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\le 1-\gamma \quad (2.7)$$

The left-hand side of the inequality (2.7) could be written as

$$\begin{aligned} Q_2(\xi,\gamma,\rho):= \ & \sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)} \\ = \ & [(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{n(n-1)(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)} \\ & +(1-\gamma)\sum_{n=2}^{\infty}\frac{n(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}, \end{aligned}$$

and from (2.2) and (2.3) we get

$$Q_2(\xi,\gamma,\rho)=[(1-\rho)\sec\xi+\rho(1-\gamma)]{E^{\theta}}''_{\alpha,\beta}(1)+(1-\gamma)\left[{E^{\theta}}'_{\alpha,\beta}(1)-1\right]$$

Hence, the assumption (2.6) implies that $Q_2(\xi,\gamma,\rho)\le 1-\gamma$ that is (2.7) holds, and consequently $E^{\theta}_{\alpha,\beta}\in\mathcal{K}(\xi,\gamma,\rho)$.

## 3. IMAGE PROPERTIES OF $\Lambda^{\alpha}_{\beta}$ Operator

Making use of the Lemma 1.1 and Lemma 1.3 we will focus the influence of the $\Lambda^{\alpha}_{\beta}$ operator for the functions of the class $\mathcal{R}^{\tau}(\vartheta,\delta)$, and we will give sufficient conditions such that these images are in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, respectively.

**Theorem 3.1.**

If

$$\frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E_{\alpha,\beta}^{\theta}(1)-1\right] \\ +(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}^{\theta}(t)}{t}-1\right)dt\leq 1-\gamma \tag{3.1}$$

then

$$\Lambda_{\beta}^{\alpha}(\mathcal{R}^{\tau}(\vartheta,\delta))\subset\mathcal{S}(\xi,\gamma,\rho).$$

Proof. Let $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$ be of the form (1.1). To prove that $\Lambda_{\beta}^{\alpha}(f)\in\mathcal{S}(\xi,\gamma,\rho)$, in view of Lemma 1.1 it is required to show that

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}|a_n|\leq 1-\gamma.$$

Let we denote the left-hand side of the above inequality by

$$Q_3(\xi,\gamma,\rho):=\textstyle\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\dfrac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}|a_n|$$

Since $f\in\mathcal{R}^{\tau}(\vartheta,\delta)$, by Lemma 1.3 we have

$$|a_n|\leq\frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)},\ n\in\mathbb{N}\setminus\{1\},$$

and using the inequality $1+\vartheta(n-1)\geq\vartheta n$ we obtain that

$$\begin{aligned} Q_3(\xi,\gamma,\rho)\leq\ & \frac{2|\tau|(1-\delta)}{\vartheta}\left\{\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\right. \\ & \left.\times\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\right\} \\ =\ & \frac{2|\tau|(1-\delta)}{\vartheta}\left\{\sum_{n=2}^{\infty}[(1-\rho)\sec\xi+\rho(1-\gamma)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\right. \\ & \left.+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\right\} \end{aligned}$$

From the above inequality, using (2.1), we get

$$\begin{aligned} Q_3(\xi,\gamma,\rho)\leq\ & \frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E_{\alpha,\beta}^{\theta}-1\right] \\ & +(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}^{\theta}(t)}{t}-1\right)dt, \end{aligned}$$

hence, the assumption (3.1) implies then $Q_3(\xi,\gamma,\rho) \le 1-\gamma$, that is $\Lambda_\beta^\alpha(f) \in \mathcal{S}(\xi,\gamma,\rho)$.

Using Lemma 1.2 and following the same procedure as in the proof of Theorem 2.2, we have the subsequent result.

**5. Theorem 3.2. If**

$$\begin{aligned}&\frac{2|\tau|(1-\delta)}{\vartheta}\Big\{[(1-\rho)\sec\xi+\rho(1-\gamma)]E^{\theta}{}'_{\alpha,\beta}(1)\\&+(1-\rho)(1-\gamma-\sec\xi)E^{\theta}_{\alpha,\beta}(1)-(1-\gamma)\Big\}\le 1-\gamma\end{aligned} \tag{3.2}$$

then

$$\Lambda_\beta^\alpha(\mathcal{R}^\tau(\vartheta,\delta)) \subset \mathcal{K}(\xi,\gamma,\rho)$$

Proof. Let $f \in \mathcal{R}^\tau(\vartheta,\delta)$ be of the form (1.1). In view of Lemma 1.2, to prove that $\Lambda_\beta^\alpha(f) \in \mathcal{K}(\xi,\gamma,\rho)$ we have to show that

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}|a_n| \le 1-\gamma \tag{3.3}$$

Since $f \in \mathcal{R}^\tau(\vartheta,\delta)$, then by Lemma 1.3 we have

$$|a_n| \le \frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)},\ n\in\mathbb{N}\setminus\{1\},$$

and $1+\vartheta(n-1)\ge\vartheta n$. Denoting the left-hand side of the inequality (3.3) by

$$Q_4(\xi,\gamma,\rho):=\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}|a_n|,$$

we deduce that

$$\begin{aligned}Q_4(\xi,\gamma,\rho) \le\ & \frac{2|\tau|(1-\delta)}{\vartheta}\sum_{n=2}^{\infty}[(1-\rho)\sec\xi(n-1)+(1-\gamma)(1+n\rho-\rho)]\\&\times\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\\=\ & \frac{2|\tau|(1-\delta)}{\vartheta}\Bigg\{[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{n(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\\&+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{\mathrm{n}!\,\Gamma(\alpha(n-1)+\beta)}\Bigg\}\end{aligned}$$

Now, by using (2.1) and (2.2), the above inequality yields to

$$
\begin{aligned}
Q_4(\xi,\gamma,\rho) \leq\ & \frac{2|\tau|(1-\delta)}{\vartheta}\Big\{[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[{E^{\theta}}'_{\alpha,\beta}(1)-1\right] \\
& +(1-\rho)(1-\gamma-\sec\xi)\left[E^{\theta}_{\alpha,\beta}(1)-1\right]\Big\} \\
=\ & \frac{2|\tau|(1-\delta)}{\vartheta}\Big\{[(1-\rho)\sec\xi+\rho(1-\gamma)]{E^{\theta}}'_{\alpha,\beta}(1) \\
& +(1-\rho)(1-\gamma-\sec\xi)E^{\theta}_{\alpha,\beta}(1)-(1-\gamma)\Big\}.
\end{aligned}
$$

Therefore, the assumption (3.2) yields to $Q_4(\xi,\gamma,\rho) \leq 1-\gamma$, which implies the inequality (3.3), that is $\Lambda_{\beta}^{\alpha}(f) \in \mathcal{K}(\xi,\gamma,\rho)$.

## 6. THE Alexander INTEGRAL OPERATOR FOR $E^{\theta}_{\alpha,\beta}$

Theorem 4.1. Let the function $\Psi_{\beta}^{\alpha}$ be given by

$$\Psi_{\beta}^{\alpha}(z) = \int_0^z \frac{E^{\theta}_{\alpha,\beta}(t)}{t}\,dt,\ z \in \mathbb{D} \quad (4.1)$$

If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]E'_{\alpha,\beta}(1)+(1-\rho)(1-\gamma-\sec\xi)E^{\theta}_{\alpha,\beta}(1) \leq 2(1-\gamma),$$

then $\Psi_{\beta}^{\alpha} \in \mathcal{K}(\xi,\gamma,\rho)$.

Proof. Since

$$\Psi_{\beta}^{\alpha}(z) = z+\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\cdot\frac{z^n}{n},\ z \in \mathbb{D}, \quad (4.2)$$

according to Lemma 1.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{1}{n}\cdot\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \leq 1-\gamma$$

or, equivalently

$$\sum_{n=2}^{\infty} [(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \leq 1-\gamma$$

Now, the proof of Theorem 4.1 is parallel to that of Theorem 2.1, and so it will be omitted.

Theorem 4.2. Let the function $\Psi_{\beta}^{\alpha}$ be given by (4.1). If

$$
\begin{aligned}
& [(1-\rho)\sec\xi+\rho(1-\gamma)]\left(E^{\theta}_{\alpha,\beta}(1)-1\right) \\
& +(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E^{\theta}_{\alpha,\beta}(t)}{t}-1\right)dt \leq 1-\gamma,
\end{aligned}
\quad (4.3)
$$

then $\Psi_\beta^\alpha \in \mathcal{S}(\xi,\gamma,\rho)$.

Proof. Since $\Psi_\beta^\alpha$ has the power series expansion (4.2), then by Lemma 1.1 it is sufficient to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \le 1-\gamma.$$

The left-hand side of the above inequality could be rewritten as

$$\begin{aligned} Q_5(\xi,\gamma,\rho) = & \sum_{n=2}^{\infty} \frac{1}{n}[(1-\rho)(n-1)\sec\xi + (1-\gamma)(1+n\rho-\rho)] \\ & \times \frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \\ = & \sum_{n=2}^{\infty} [(1-\rho)\sec\xi + \rho(1-\gamma)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \\ & +(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\cdot\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \end{aligned}$$

and using (2.1) we get

$$\begin{aligned} Q_5(\xi,\gamma,\rho) \le & [(1-\rho)\sec\xi + \rho(1-\gamma)]\left[E_{\alpha,\beta}^{\theta}(1)-1\right] \\ & +(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}^{\theta}(t)}{t}-1\right)dt. \end{aligned}$$

Therefore, if the assumption (4.3) holds, then $Q_5(\xi,\gamma,\rho) \le 1-\gamma$. Hence, $\Psi_\beta^\alpha \in \mathcal{S}(\xi,\gamma,\rho)$

Remark 4.1. By taking $\rho = 0$ in Theorems 2.1-4.2, we can easily attain the sufficient condition for $E_{\alpha,\beta}^{\theta} \in \mathcal{S}(\xi,\gamma)$ and $E_{\alpha,\beta}^{\theta} \in \mathcal{K}(\xi,\gamma)$. The function $E_{\alpha,\beta}^{\theta}$ is associated with Mittag-Leffler functions and has not been studied sofar. We left this as an exercise to interested readers.

For the special case $\alpha = 1/2$ and $\beta = 1, \theta = 1$, that is connected with the error function can derive some results based on the error function. Thus, a simple computation shows that if

$$\mathcal{E}(z) := E_{\frac{1}{2},1}(z) = \sum_{n=1}^{\infty}\frac{z^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

then

$$\mathcal{E}(1)=\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)},\ \mathcal{E}'(1)=\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)},\ \mathcal{E}''(1)=\sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)}$$

$$\int_0^1\left(\frac{\mathcal{E}(t)}{t}-1\right)dt=\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)},$$

$$\mathcal{L}:=\Lambda_1^{\frac{1}{2}}f(z)=f(z)*\mathcal{E}(z)=z+\sum_{n=2}^{\infty}\frac{a_n z^n}{\Gamma\left(\frac{n+1}{2}\right)},\ (4.4)$$

$$\mathcal{P}:=\Psi_1^{\frac{1}{2}}(z)=\int_0^z\frac{\mathcal{E}(t)}{t}dt=\sum_{n=1}^{\infty}\frac{z^n}{n\Gamma\left(\frac{n+1}{2}\right)}\ (4.5)$$

Using the above relations, from Theorems 2.1 and 2.2 we get, respectively. Example 4.1. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$<2(1-\gamma)$$

then $\mathcal{E}\in\mathcal{S}(\xi,\gamma,\rho)$.

Example 4.2. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{n(n-1)}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\gamma)\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}\leq 2(1-\gamma),$$

then $\mathcal{E}\in\mathcal{K}(\xi,\gamma,\rho)$.

Similarly, Theorems 4.1 and 4.2 give us the next examples.

Example 4.3. If

$$\frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}\leq 1-\gamma,$$

then

$$\mathcal{L}(\mathcal{R}^{\tau}(\vartheta,\delta))\subset\mathcal{S}(\xi,\gamma,\rho),$$

where $\mathcal{L}$ is defined by (4.4).

Example 4.4. If

$$\frac{2|\tau|(1-\delta)}{\vartheta}\left\{[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}\right.$$
$$\left.+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}-(1-\gamma)\right\}\leq 1-\gamma$$

then

$$\mathcal{L}(\mathcal{R}^{\tau}(\vartheta,\delta))\subset\mathcal{K}(\xi,\gamma,\rho)$$

where $\mathcal{L}$ is defined by (4.4).

Finally, from Theorems 4.1 and 4.2 we have the following.

Example 4.5. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=1}^{\infty}\frac{n}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=1}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leqslant 2(1-\gamma),$$

then $\mathcal{P}\in\mathcal{K}(\xi,\gamma,\rho)$, where $\mathcal{L}$ is defined by (4.5). Example 4.6. If

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\frac{1}{\Gamma\left(\frac{n+1}{2}\right)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n\Gamma\left(\frac{n+1}{2}\right)}$$
$$\leq 1-\gamma,$$

then $\mathcal{P}\in\mathcal{S}(\xi,\gamma,\rho)$, where $\mathcal{L}$ is defined by (4.5).

## 7. CONCLUSIONS

In this investigation we obtained sufficient conditions and inclusion results for functions $f\in\mathcal{A}$ to be in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, and information regarding the images of functions by applying convolution operator with Mittag-Leffler functions.

The investigation methods are based on some recent results and techniques found in [15] and [16], and we determined sufficient conditions for the functions $E_{\alpha,\beta}$ to belongs to the new defined classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$.

Moreover, we found sufficient conditions such that the images of the functions belonging to the class $\mathcal{R}^{\tau}(\vartheta,\delta)$ by the new defined convolution operator $\Lambda_{\beta}^{\alpha}$ are in the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$, respectively.

Finally, we determined sufficient conditions such that the functions $\Psi_{\beta}^{\alpha}$ obtained as images of $E_{\alpha,\beta}$ via the Alexander integral operator belong to the classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$

We emphasize that till now such kind of results doesn't appeared in any previous articles: the general classes $\mathcal{S}(\xi,\gamma,\rho)$ and $\mathcal{K}(\xi,\gamma,\rho)$ are completely new and introduced in [15,16], while any type of such results were not studied previously.

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